2018 | **HSC Trial Examinations**

Centre Number

Student Number

Killara HIGH SCHOOL
MATHEMATICS DEPARTMENT

Mathematics Extension 1 Trial

General Instructions

- Reading time – 5 minutes
- Working time **2** hours
- Write using blue or black pen
- NESA approved calculators may be used
- A formulae and data sheet is provided
- For questions 11 to 14, show relevant mathematical reasoning and/or calculations

Section I - 10 marks (Pages 3 – 6) • Attempt Questions 1 to 10 • Allow about 15 minutes for this section	Multiple Choice Questions	**/10**
Section II - 60 marks (Page 6 – 11) • Attempt Questions 11 to 14 • Allow about 1h 45 minutes for this section	Question 11	**/15**
	Question 12	**/15**
	Question 13	**/15**
	Question 14	**/15**
	Total	**/70**

THIS QUESTION PAPER MUST NOT BE REMOVED FROM THE EXAMINATION ROOM

This assessment task constitutes 40% of the HSC *Course Assessment*

Section I

10 marks
Attempt Questions 1 to 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for questions 1 to 10 (Detach from paper)

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1. Which of the following is a factor of $x^3 + 2x^2 - 7x + 4$?

   (A) $x - 1$

   (B) $x + 2$

   (C) $x - 2$

   (D) $x + 1$

2. Given that $N = 50 + 40e^{kt}$, which expression is equal to $\frac{dN}{dt}$?

   (A) $k(50 - N)$

   (B) $k(40 - N)$

   (C) $k(N - 50)$

   (D) $k(N - 40)$

3. If $A = \tan^{-1}(x)$, then the value of $sin2A$ is:

   (A) $\frac{2x}{1-x^2}$

   (B) $\frac{2x}{\sqrt{1-x^2}}$

   (C) $\frac{2x}{\sqrt{1+x^2}}$

   (D) $\frac{2x}{1+x^2}$

4. The domain and range of the function $f(x) = 2\cos^{-1}\left(\frac{x}{3}\right)$ is given by:

   (A) $0 \leq x \leq 3; \ -\pi \leq y \leq \pi$.

   (B) $-3 \leq x \leq 3; \ 0 \leq y \leq \pi$

   (C) $-3 \leq x \leq 3; \ 0 \leq y \leq 2\pi$

   (D) $-\pi \leq x \leq \pi; \ 0 \leq y \leq 2$

5. What is the x-intercept of the normal to the parabola $x^2 = 4ay$ at the point $(2ap, ap^2)$ on the parabola?

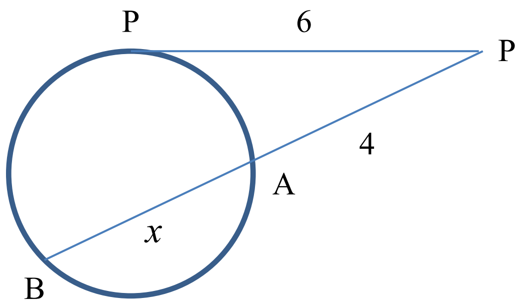
(A) $ap(p^2 + 1)$

(B) $ap(p^2 + 2)$

(C) ap^2

(D) $-ap^2$

6. A secant drawn from P to B and a tangent drawn from P to T meet the circle as shown below.



What is the value of the pronumeral x ?

(A) $x = 9$

(B) $x = \frac{3}{2}$

(C) $x = 2$

(D) $x = 5$

7. Find $\int \frac{e^{2x}}{1 + e^{2x}}\ dx$

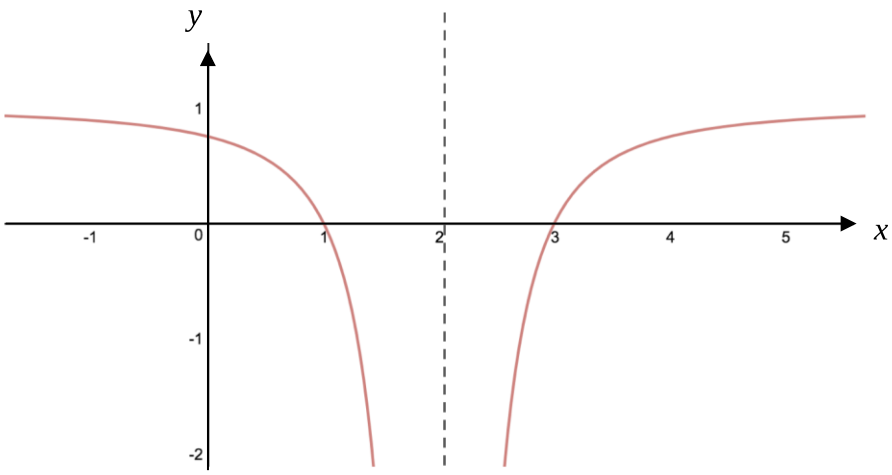
(A) $\tan^{-1}(e^x)$

(B) $2\tan^{-1}(e^{2x})$

(C) $\frac{1}{2}\ln(1 + e^{2x})$

(D) $2\ln(1 + e^{2x})$

8.



Find the function $y = f(x)$ whose graph is given above:

(A) $f(x) = \frac{(x-1)(x-3)}{(x-2)^2}$

(B) $f(x) = \frac{2(x-1)(x-3)}{x-2}$

(C) $f(x) = \frac{2(x-1)(x-3)}{(x-2)^3}$

(E) $f(x) = \frac{2(x-1)(x-3)}{(x-2)^2}$

9.

The graph of $f(x) = 0.6\cos^{-1}(x-1)$, defines a curve that, when rotated about the y-axis will produce a solid that is to be the shape and size of a new biscuit.
Which integral expression will give the volume of the biscuit?

(A) $\pi\int_0^{0.6}\left[cos\left(\frac{3}{5}y\right)+1\right]^2 dy$

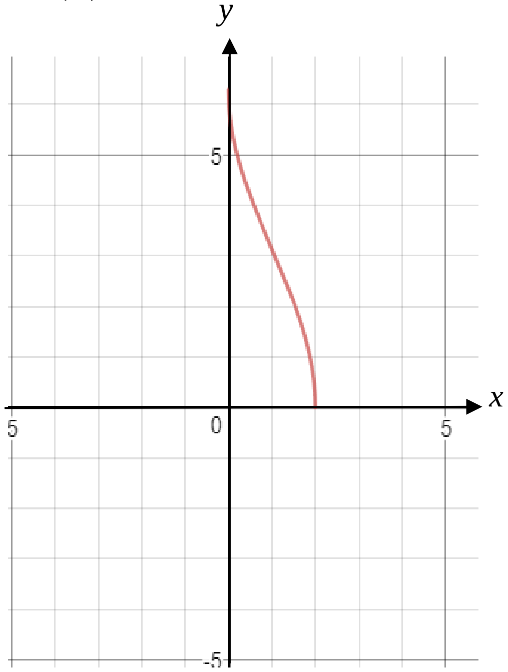
(B) $\pi\int_0^{0.6}\left[cos\left(\frac{5}{3}y\right)+1\right]^2 dy$

(C) $\pi\int_0^{0.6\pi}\left[cos\left(\frac{5}{3}y\right)+1\right]^2 dy$

(D) $\pi\int_0^{0.6\pi}\left[cos\left(\frac{3}{5}y\right)+1\right]^2 dy$

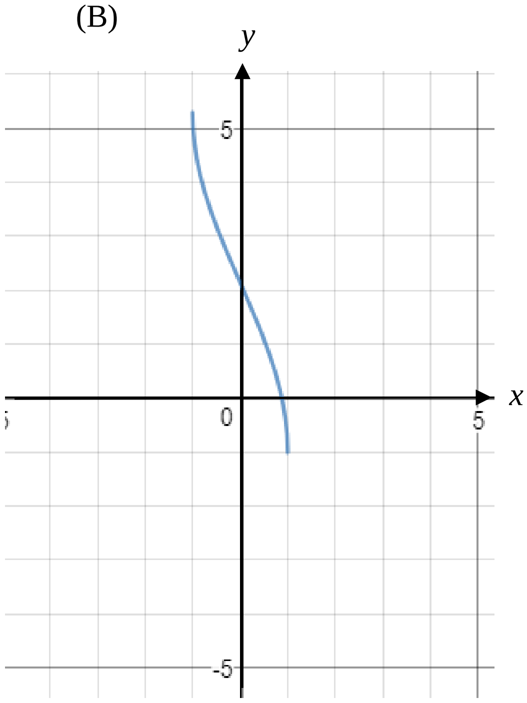
10. Which of the graphs match with the given equation?

(A)



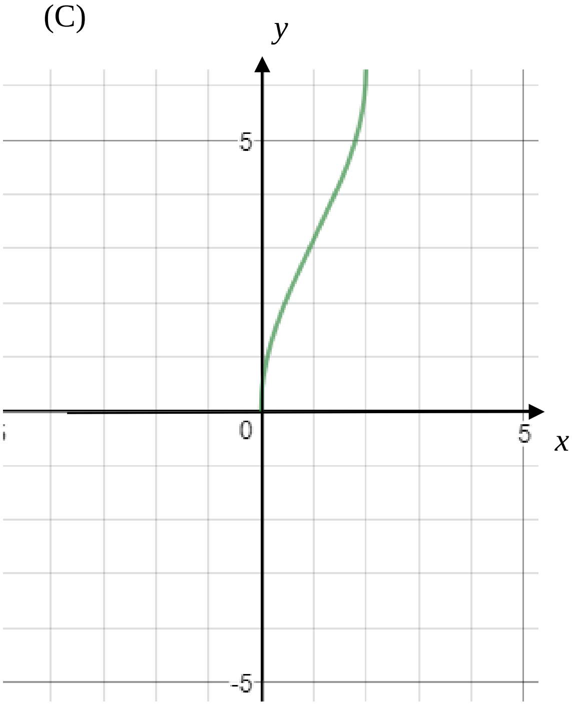
$y = 2\cos^{-1}(x + 1)$

(B)



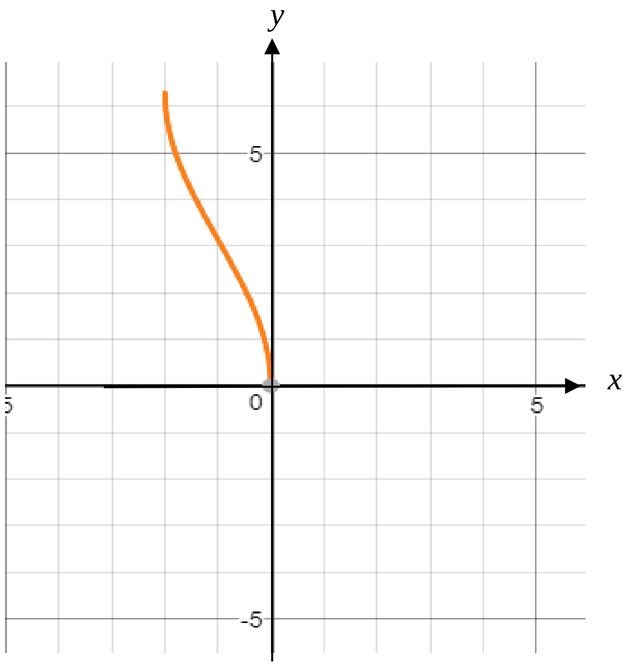
$y = 2\cos^{-1}(x - 1)$

(C)



$y = 2\cos^{-1}(1 - x)$

(D)



$y = 2\cos^{-1}x - 1$

Section II
90 marks
Attempt Questions 11 to 14
Allow about 1 hours and 45 minutes for this section

Answer each question in a **separate writing booklet**. Extra writing booklets are available.

In Questions 11 to 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) Evaluate $\int_0^{\frac{\pi}{4}} \cos^2 2x \, dx$ **3**

(b) The acute angle between $5x - y - 3 = 0$ and $y = mx + 4$ is 45^o. Find two possible values of *m*. **2**

(c) Solve $\frac{2x}{x-2} \geq 1$ **3**

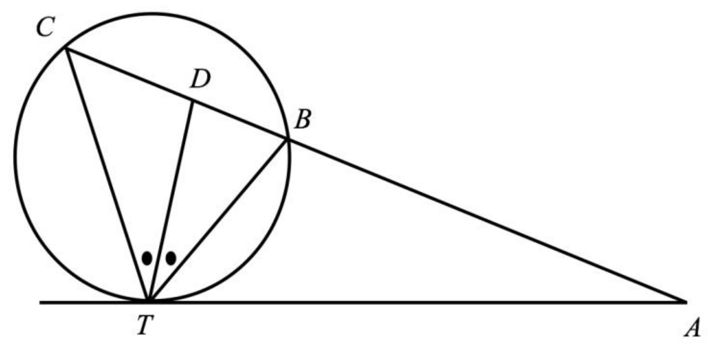
(d) Express $\cos x + \sin x$ in the form $R\cos(x+\alpha)$, where α is in radians. **2**

(e) Evaluate $\int_1^2 \frac{4x}{\sqrt{x^2-1}} dx$ using the substitution $u = x^2 - 1$. **3**

(f) Let α, β, γ be roots of $P(x) = x^3 - 2x^2 - x + 2$. Evaluate $\alpha^2 + \beta^2 + \gamma^2$ given that $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$ **2**

Question 12 (15 marks) Use a SEPARATE writing booklet

(a)

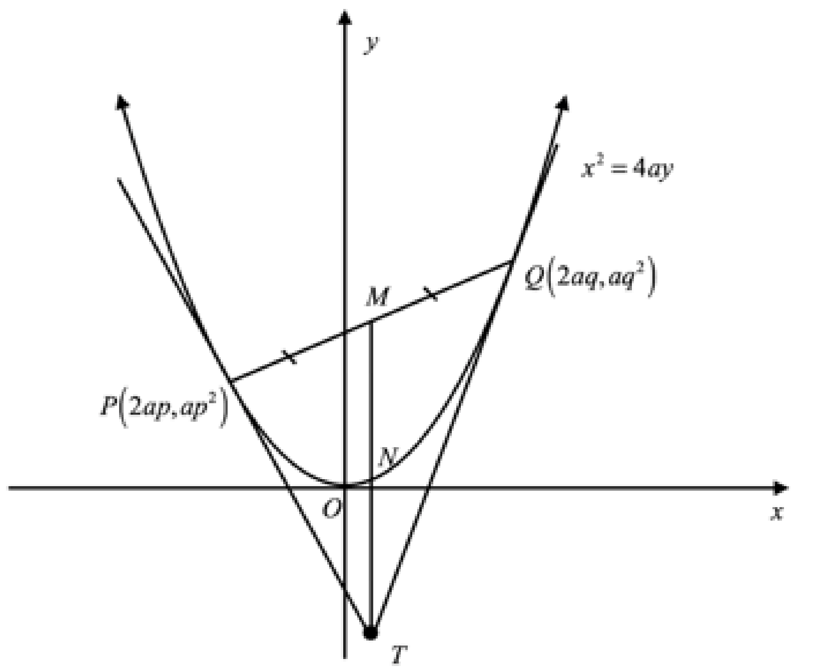


3

TA is a tangent to a circle. Line *ABDC* intersects the circle at *B* and *C*. Line *TD* bisects angle *BTC*.

Prove $AT = AD$

(b)



In the diagram, $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are two points on the parabola $x^2 = 4ay$.

i) Show that the tangent at *P* has equation $y = px - ap^2$. **2**

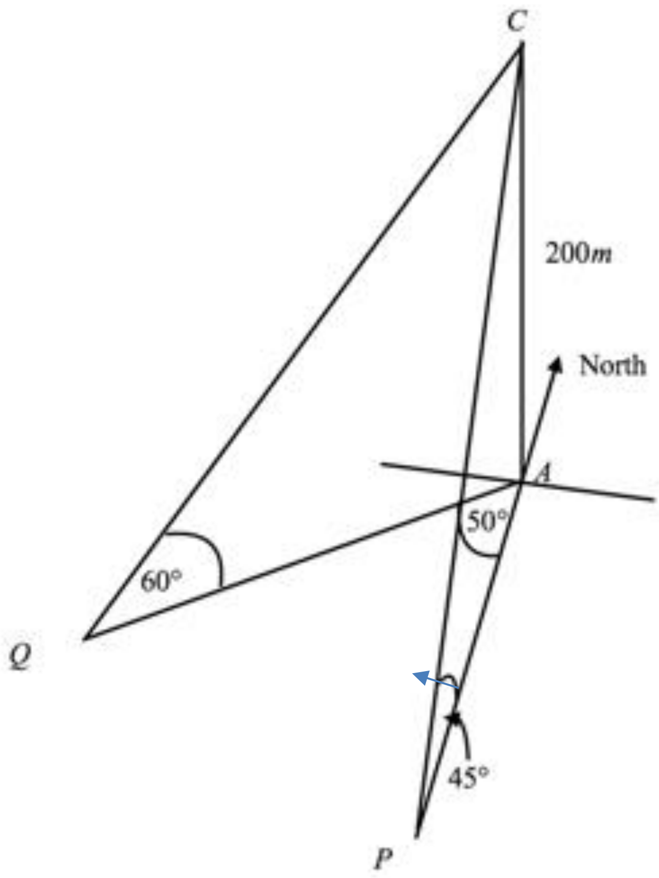
ii) The tangent at *P* and *Q* meet at *T*. Assuming that the tangent at *Q* is $y = qx - aq^2$, show that *T* is the point $(a(p+q), apq)$. **2**

iii) *M* is the midpoint of the chord *PQ*. Show that *MT* is parallel to the axis of symmetry of the parabola. **2**

Question 12 continues on page 8

Question 12 continued

(c) From the top, C, of a vertical cliff, 200 m high, two ships P and Q are observed at sea level. A is the foot of the cliff at sea level. P is due south of A and the angle of elevation of C from P is 45°. Q is $S\ 50°\ W$ of A and the angle of elevation of C from Q is 60°. **3**



Find the distance PQ (to the nearest metre).

(d) i) By considering the graph of $y = e^x$, show that the equation $e^x + x + 1 = 0$ has only one real root and that this root is negative. **2**

ii) Taking $x = -1.5$ as a first approximation to this root, use one application of Newton's method to find a better approximation. (Give your answer correct to three decimal places) **2**

Question 13 (15 marks) Use a SEPARATE writing booklet

(a) Use the t-formula to solve for x (to the nearest minute). **3**

$$7sinx - 4cosx = 4, \qquad \text{for } 0^o \leq x \leq 360^o$$

(b) Prove by mathematical induction that **3**

$$\frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \text{........} + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}$$

For all integers $n \geq 1$.

(c) Consider the function

$$f(x) = 2\sin^{-1}\sqrt{x} - \sin^{-1}(2x - 1)$$

i) Find the domain of $f(x)$ **1**

ii) Show that $f'(x) = 0$ **3**

iii) Sketch the graph of $y = f(x)$ **1**

(d) The rise and fall of a tide can be modelled as simple harmonic motion. A cruise ship needs 11 metres of water to pass down a channel safely. At low tide the channel is 8 metres deep and at high tide it is 12 metres deep. Low tide is at
10 am and high tide is 4pm.

(i) Show the water depth, y metres, in the channel, is given by **2**

$$y = 10 - 2\cos\left(\frac{\pi t}{6}\right)$$

(ii) Find the earliest time period after 10am (i.e. between which two times) that the cruise ship can safely proceed through the channel. **2**

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) On a tropical island that is being set up as a nature reserve, there are initially 100 nesting pairs of seagulls. In the 1st year, the number of pairs increases by 8. One theory suggested that the number N of nesting pair after t years would satisfy the differential equation $\frac{dN}{dt} = \frac{1}{k}N(500 - N)$.

i) Show that the value of k is 5000. **1**

ii) Given that $\frac{10a}{x(a-x)} = \frac{10}{x} + \frac{10}{a-x}$ show that the differential equation can be expressed in the form $\frac{dt}{dN} = \frac{10}{N} + \frac{10}{500-N}$ and find the equation for t in term of N. **3**

iii) Predict, using your answer in part (ii), after how many years (to the nearest year) the number of pairs of nesting seagulls on the island will be 300. **1**

(b) i) The acceleration of a particle moving in the x −axis is numerically equal to $a = v^2(1 - v)cm\ s^{-2}$.
Initially $x = 0\ cm, v = 0.5cm\ s^{-1}$.

Show that $v = \frac{e^x}{1+e^x}$ (Hint: $\frac{1}{v(1-v)} = \frac{1}{v} + \frac{1}{1-v}$) **2**

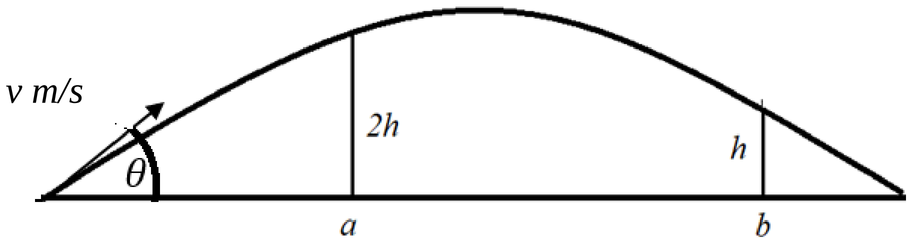
ii) Show that $t = x + 1 - e^{-x}$. **2**

iii) Hence determine the range of values of x and v. **2**

Question 14 continues on page 10

(c) **Question 14 continued**

A particle is projected at an angle θ to the horizontal with a velocity of Vms^{-1} so as to just clear two walls. The walls of height $2h$ metres and h metres are at distances a metres and b metres respectively from the point of projection.



i) Given that $x = Vcos\theta\ t$ and $y = Vsin\theta\ t - \frac{1}{2}gt^2$, where x is the horizontal displacement and y is the horizontal displacement in t seconds, show that $y = xtan\theta - \frac{gx^2}{2V^2}(1 + tan^2\theta)$ **1**

ii) Hence show that $tan\theta = \frac{h(2b^2-a^2)}{ab(b-a)}$ **3**

END OF EXAMINATION

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2018 EX1 Task 3 solution

Q1 A Q2 C Q3 D Q4 (C) Q5 B Q6 D Q7 C Q8 A Q9 C Q10 C

1.	$P(1) = 1^3 + 2\times 1^2 - 7\times 1 + 4$ $= 1+2-7+4 = 0$ $\therefore (x-1)$ is a factor $\therefore$ (A)	**5.**	$x + py = 2ap + ap^3$ $\therefore y = 0,\ x = 2ap + ap^3$ B
2.	$N = 50 + 40.e^{kt}$ $\frac{dN}{dt} = 40k.e^{kt}$ $= k.(40e^{kt})$ as $N = 50 + 40e^{kt}$ $40e^{kt} = (N-50)$ $\therefore \frac{dN}{dt} = k(N-50)$ $\therefore$ (C)	**6.**	$TP^2 = PA.PB$ $6^2 = 4(4+x)$ $36 = 16 + 4x$ $\therefore 4x = 20$ $\therefore x = 5 \quad \therefore$ (D)
		7.	C
		8.	A
		9.	$y = \frac{\pi}{4} - \tan^{-1}\frac{x}{3}$ R: $-\frac{\pi}{4} < y < \frac{3\pi}{4}$ (C)
3.	$sin2A = 2sinAcosA = \frac{2}{\sqrt{x^2+1}}.\frac{2}{\sqrt{x^2+1}}. = \frac{2x}{1+x^2}.$ (D)	**10.**	$y = \frac{3}{5}\cos^{-1}(x-1)$ $\therefore x = \cos\left(\frac{5}{3}y\right) + 1$ $0 \le \frac{5}{3}y = \cos^{-1}(x-1) \le \pi$ $\therefore 0 \le y \le \frac{3}{5}\pi$ C
4.	(C)		

Q 11 a

$$\int_0^{\frac{\pi}{4}} \cos^2 2x \, dx$$

as $\cos 2x = 2\cos^2 x - 1$

$\Rightarrow \cos 4x = 2\cos^2(2x) - 1$

$\therefore 2\cos^2(2x) = \cos 4x + 1$

$\cos^2 2x = \frac{1}{2}(\cos 4x + 1)$

$$\therefore \int_0^{\frac{\pi}{4}} \cos^2 2x \, dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} (\cos 4x + 1) \, dx$$ [1 mark]

$$= \frac{1}{2}\left[\frac{1}{4}\sin 4x + x\right]_0^{\frac{\pi}{4}}$$ [1 mark]

$$= \frac{1}{2}\left[\left(\frac{1}{4}\sin\pi + \frac{\pi}{4}\right) - \left(\frac{1}{4}\sin 0 + 0\right)\right]$$ note $\sin\pi = 0$

$$= \frac{\pi}{8}$$ [1 mark]

11 b

$5x - y - 3 = 0$ $\qquad$ $y = mx + 4$

$y = 5x - 3$

$\therefore m_1 = 5$ $\qquad$ $\therefore m_2 = m$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 . m_2}\right|$$

$$\tan 45° = \left|\frac{5 - m}{1 + 5m}\right|$$

$$\therefore 1 = \left|\frac{5 - m}{1 + 5m}\right|$$ [1 mark]

$$1 = \frac{5-m}{1+5m} \qquad -1 = \frac{5-m}{1+5m}$$

$1 + 5m = 5 - m$

$\therefore 6m = 4$

$\therefore m = \frac{2}{3}$

$-1 - 5m = 5 - m$

$\therefore -6 = 4m$

$\therefore m = \frac{-3}{2}$

$\therefore m = \frac{2}{3}, \frac{-3}{2}$

[1 mark for both values of m]

11c

Solve $\frac{2x}{x-2} \geq 1$ note $x \neq 2$

$(x-2)^{2} \cdot \frac{2x}{(x-2)} \geq 1(x-2)^2$ [1 mark]

$(x-2)2x \geq (x-2)^2$

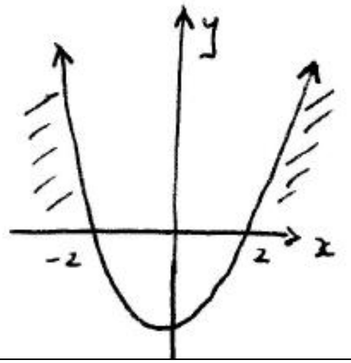
$0 \geq (x-2)^2 - (x-2)2x$

$\therefore \; 0 \geq (x-2)[(x-2)-2x]$

$0 \geq (x-2)(-x-2)$

$\left(0 \geq (x-2) \cdot -1(x+2) \right) \times -1$

$(x-2)(x+2) \geq 0$ [1 mark for correct rearranging & simplifying]



$\therefore \; x \leq -2, \; x > 2$ noting $x \neq 2$. } [1 mark]

11d $\cos x + \sin x = R\cos(x+\alpha)$

<u>Note</u>: $\cos(x+\alpha) = \cos x\cos\alpha - \sin x\sin\alpha$

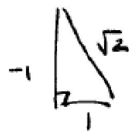
$\therefore 1\cos x + 1\sin x = R\cos x\cos\alpha - R\sin x\sin\alpha$

$\therefore R\cos\alpha = 1$

$R\sin\alpha = -1$

$\therefore \dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{-1}{1} \Rightarrow \tan\alpha = -1$

$\therefore \alpha = \tan^{-1}(-1) = \dfrac{-\pi}{4}$ [1 mark]



$R = \sqrt{(1)^2+(-1)^2} = \sqrt{2}$ (By Pythagoras') [1 mark for R]

$\therefore \cos x + \sin x = \sqrt{2}\cos\left(x - \dfrac{\pi}{4}\right)$

11e

$$\int_1^2 \frac{4x}{\sqrt{x^2-1}}\,dx \quad \text{using } u = x^2-1$$

if $u = x^2 - 1$

$$\frac{du}{dx} = 2x$$

$\therefore du = 2x\,dx$

terminals $\left.\begin{array}{l} u = 2^2-1 = 3 \\ u = 1^2-1 = 0 \end{array}\right\}$ [1 mark, terminals and $\frac{du}{dx}$]

$$\therefore \int_1^2 \frac{4x}{\sqrt{x^2-1}}\,dx$$

$$= 2\int_1^2 \frac{2x}{\sqrt{x^2-1}}\,dx$$

$$= 2\int_0^3 \frac{1}{\sqrt{u}}\,du$$

$$= 2\int_0^3 u^{-1/2}\,du$$

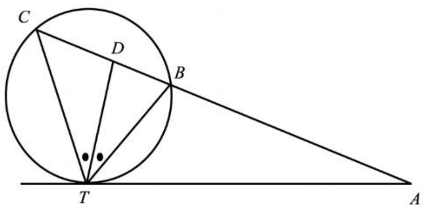
$= 4\left[u^{1/2}\right]_0^3$ [1 mark]

$= 4\left(\sqrt{3} - 0\right)$

$= 4\sqrt{3}$ [1 mark]

11 f $P(x) = x^3 - 2x^2 - x + 2$ $\alpha + \beta + \gamma = \frac{-b}{a} = 2$ $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = -1$ } [1 mark] $\therefore \alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$ $= 2^2 - 2 \times -1$ $= 4 + 2$ $= 6$ [1 mark]		

(a) **3**



TA is a tangent to a circle. Line *ABDC* intersects the circle at *B* and *C*. Line *TD* bisects angle *BTC*.

Prove $AT = AD$

3 marks for conclusion with reasoning and working 2 marks for significant progress to solution, finding multiple other angles with reasoning, or equivalent

let $\angle ATB = \alpha$
let $\angle BTD = \beta$
$\angle ACT = \alpha$ (angle between a tangent and a chord is equal to the angle on the circumference subtended by the chord in the alternate segment)

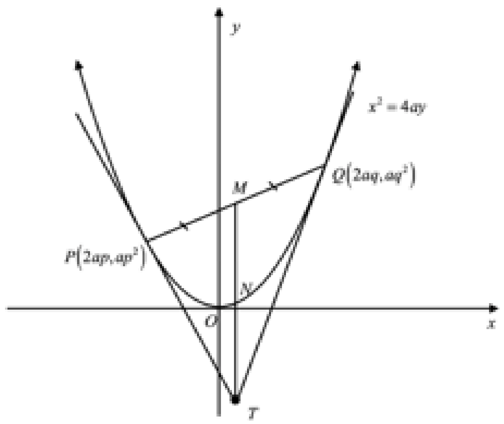
$\therefore \angle CAT = 180 - (\alpha + 2\beta) - \alpha$ (angle sum of a triangle)
similarly,

$$\angle ADT = 180 - \angle CAT - (\alpha + \beta)$$
$$= 180 - 180 + (\alpha + 2\beta) + \alpha - (\alpha + \beta)$$
$$= \alpha + \beta$$

$\therefore \angle DTA = \alpha + \beta = \angle ADT$

$\therefore \triangle DTA$ is isosceles as 2 angles re equal
$\therefore AT = AD$ (angles opposite equal sides are equal in isosceles triangle)

(b)



(i)

2 marks for correct working towards solution

1 mark for differentiating the parabola or equivalent

$$x^2 = 4ay$$
$$y = \frac{x^2}{4a}$$
$$y' = \frac{x}{2a}$$

at point $P(2ap, ap^2)$ $y' = p$

$$y - y_1 = m(x - x_1)$$
$$y - ap^2 = p(x - 2ap)$$
$$y = px - 2ap^2 + ap^2$$
$$y = px - ap^2$$

as required

(ii)

2 marks for correct and complete working

1 mark for solving simultaneously to find x or y

we have:

$$y = px - ap^2$$
$$y = qx - aq^2$$

$$px - ap^2 = qx - aq^2$$
$$px - qx = ap^2 - aq^2$$
$$x(p - q) = ap^2 - aq^2$$
$$x(p - q) = a(p - q)(p + q)$$
$$x = a(p + q)$$
$$y = p\big(a(p + q)\big) - ap^2$$
$$= ap^2 + apq - ap^2$$
$$= apq$$
$$\therefore \big(a(p + q), apq\big)$$

(iii)

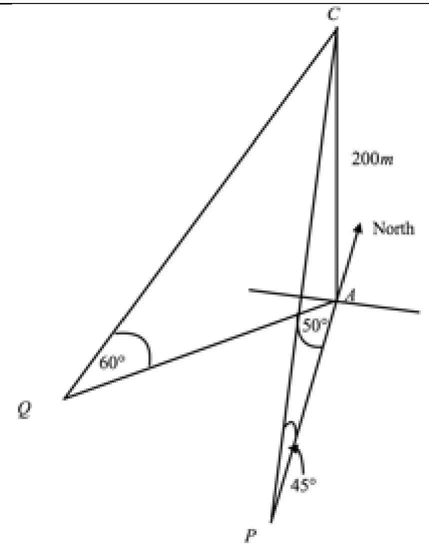
2 marks for correct answer with sound reasoning

1 mark for finding the x value of M or equivalent

x value given by

$$x = \frac{2ap + 2aq}{2}$$
$$= a(p + q)$$

As *M* and *T* have the same *x* value, the line *MT* is vertical, as is the axis of the parabola, hence parallel.



3 marks for correct answer with correct and complete working

2 marks finding expressions for PA and QA and an attempt to use them

$$\tan 45^o = \frac{200}{PA} \qquad PA = \frac{200}{\tan 45^o} \qquad PA = 200$$

$$\tan 60^o = \frac{200}{QA} \qquad QA = \frac{200}{\tan 60^o} \qquad QA = \frac{200}{\sqrt{3}}$$

by the cosine rule,

$$QP^2 = PA^2 + QA^2 - 2.PA\,.\,QA\,.\cos 50^o$$
$$QP^2 = 23644.24537...$$
$$QP = 153.766854...$$

therefore QP has length 154 metres

(d) i)

2 marks for graph with both lines and clearly one intercept

1 mark for either line drawn correctly on scaled axes, or equivalent

sketch both $y = e^x$ and $y = -x - 1$

We can see there is only one intercept, hence only one solution

(ii)

2 marks for correct answer with correct working

1 mark for correct substitution or correct derivative, but not both

$y = e^x + x + 1$

$y' = e^x + 1$

by Newton's Method

$$x_2 = x_1 - \frac{f(x)}{f'(x)}$$

$$x_2 = -1.5 - \frac{e^{-1.5} - 1.5 + 1}{e^{-1.5} + 1}$$

$x_2 = -1.273638286...$

$x_2 = -1.274(3dp)$

Q13

(a) $\frac{14t}{1+t^2} - \frac{4(1-t^2)}{1+t^2} = 4$ $14t - 4 + 4t^2 = 4 + 4t^2$ $14t = 8 \quad \rightarrow \quad t = \frac{4}{7}$ $\tan\left(\frac{\theta}{2}\right) = \frac{4}{7} \quad \rightarrow \quad \frac{\theta}{2} =$ Test for $\theta = 180^o$: LHS= $7 \times 0 + 4 \times 1 = 4 = RHS$ $\therefore \theta = 180^o, \ldots \ldots$	1 mark-using correct formula 1 mark – correct answer of θ. 1 mark – testing for solution 180°	
(b) Prove that it is true for $n = 1$ LHS = $\frac{1}{1\times2} = \frac{1}{2}$ RHS = $1 - \frac{1}{2} = \frac{1}{2}$ LHS = RHS $\therefore$ it is true for $n = 1$	1 mark – proving n = 1	
Assume that it is true for $n = k$ i.e. $\frac{1}{1\times2} + \frac{1}{1\times3} + \frac{1}{1\times4} + \ldots\ldots\ldots + \frac{1}{k(k+1)} = 1 - \frac{1}{k+1}$		
RTP that it is true for $n = k + 1$ i.e. proving $\frac{1}{1\times2} + \frac{1}{1\times3} + \frac{1}{1\times4} + \ldots\ldots\ldots + \frac{1}{k(k+1)} + \frac{1}{(k+2)(k+1)} = 1 - \frac{1}{k+2}$ (3) By assumption $\frac{1}{1\times2} + \frac{1}{1\times3} + \frac{1}{1\times4} + \ldots\ldots\ldots + \frac{1}{k(k+1)} + \frac{1}{(k+2)(k+1)} = 1 - \frac{1}{k+1} + \frac{1}{(k+2)(k+1)}$ RHS = $1 - \frac{(k+2)-1}{(k+1)(k+2)}$ $= 1 - \frac{(k+1)}{(k+1)(k+2)}$ $= 1 - \frac{1}{k+2}$ = RHS of (3)	2 – correct steps to achieve LHS = RHS 1 – working toward answer.	
$\therefore$ If it is true for $n = k$ then it is true for $n = k + 1$ Hence the statement is true for $n = 1$ and it is true for $n = 2$ and so on. By Mathematical Induction it is true for all integer n > 0.	-1 – if no conclusion or conclusion does not make sense.	

(C)

(i) Domain: $0 \le x \le 1$ and $-1 \le 2x - 1 \le 1$

$0 \le x \le 1$ and $0 \le x \le 1$

$\therefore \; 0 \le x \le 1$

1 marks – correct domain with working

(ii) $\frac{d}{dx} f(x) = \frac{2}{\sqrt{1-x}} \times \frac{1}{2\sqrt{x}} - \frac{1}{\sqrt{1-(2x-1)^2}} \times 2$

$= \frac{1}{\sqrt{x}.\sqrt{1-x}} - \frac{2}{\sqrt{4x(1-x)}}$

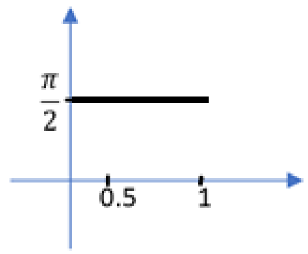
$= \frac{1}{\sqrt{x}.\sqrt{1-x}} - \frac{1}{\sqrt{x}.\sqrt{1-x}} = 0$

2 marks – correct differentiation of the two functions.

1 mark – simplify to achieve correct answer.

(iii) When $x = 0$, $f(0) = 2\sin^{-1}\sqrt{0} - \sin^{-1}(-1)$

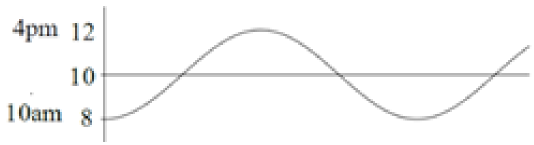
$= \frac{\pi}{2}$



1 mark – correct graph with labelling as shown.

(D)

(i)



From the diagram: centre of motion $= \frac{12+8}{2} = 10$

Amplitude is 2

Period: 10am $\rightarrow$ $4pm$ is 6 hours ie $\frac{1}{2}$ period.

$\therefore 2 \times 6 = \frac{2\pi}{n} \quad \rightarrow n = \frac{\pi}{6}$

The equation of motion is $y = 10 - 2\cos\left(\frac{\pi t}{6}\right)$

2 marks – correct answer with diagram and working shown.

1mark – correct amplitude & period.

(ii) When $y = 11 \quad \rightarrow \quad 11 = 10 - 2\cos\left(\frac{\pi t}{6}\right)$

$\cos\left(\frac{\pi t}{6}\right) = -\frac{1}{2}$

$\frac{\pi t}{6} = \frac{2\pi}{3}; \frac{4\pi}{3} \quad \rightarrow \quad t = 4 \; and \; 8$ hours

$\therefore$ it is between 2 pm and 6pm.

1 – correct equation of motion

1 – correct answers

Q9 (a)

When $n = 100, \frac{dN}{dt} = 8$

$\rightarrow \quad 8 = \frac{1}{k} \times 100(500 - 100)$

$8 = \frac{40000}{k} \quad \rightarrow \quad k = 5000$

(ii) $\frac{10k}{x(k-x)} = \frac{10}{x} + \frac{10}{k-x}$

When $k = 500$ and $x = N$

$\frac{5000}{N(500-N)} = \frac{10}{N} + \frac{10}{500-N}$

(ie) $\frac{dt}{dN} = \frac{10}{N} + \frac{10}{500-N}$

Integrating both sides w.r.t N,

$\int \frac{dt}{dN}\, dN = \int \left(\frac{10}{N} + \frac{10}{500-N}\right) dN$

$t = 10 \ln\left(\frac{N}{500-N}\right) + C$

When $t = 0$, $N = 100$

$\Rightarrow \quad 0 = 10 \ln\left(\frac{100}{400}\right) + C$

$C = 10 \ln(4)$

$\Rightarrow \quad t = 10 \ln\left(\frac{4N}{500-N}\right)$

(iii) When $N = 300$,

$t = 10 \ln\left(\frac{4 \times 300}{500-300}\right)$

$= 10 \ln(6)$ years

$\approx$

(b)

(i) $a = v\frac{dv}{dx} = v^2(1-v)$

$\Rightarrow \int \frac{1}{v(1-v)}\,dv = \int dx$

$\int \frac{1}{1-v} + \frac{1}{v}\,dv = \int dx$

$\ln\left(\frac{v}{1-v}\right) = x + c$

$x=0,\ v=0.5$

$\therefore c = 0$

$\Rightarrow \ln\left(\frac{v}{1-v}\right) = x$

$\frac{v}{1-v} = e^x$

$v = \frac{e^x}{1+e^x}$ //

(ii) $\frac{dx}{dt} = \frac{e^x}{1+e^x}$

$\int \frac{1+e^x}{e^x}\,dx = \int dt$

$-e^{-x} + x = t + c$

$t=0,\ x=0$

$\therefore c = -1$

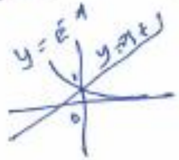
$\Rightarrow t = 1 + x - e^{-x}$

(iii) as $t \geq 0$

$1 + x - e^{-x} \geq 0$

$\Rightarrow x + 1 \geq e^{-x}$

$\Rightarrow x \geq 0$



$v = \frac{e^x}{1+e^x}$

$= \frac{1}{e^{-x}+1}$

as $x \geq 0$

$\frac{1}{2} \leq \frac{1}{e^{-x}+1} < 1 \Rightarrow \frac{1}{2} \leq v < 1$

(c) (i) $x = V\cos\theta \cdot t \Rightarrow t = \frac{x}{V\cos\theta}$

Sub in $t = \frac{x}{V\cos\theta}$ into y

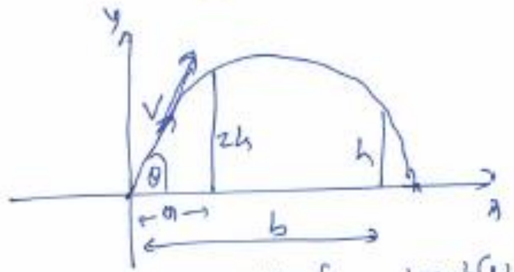
$$y = V\sin\theta\left(\frac{x}{V\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{V\cos\theta}\right)^2$$

$$= x\tan\theta - \frac{gx^2}{2V^2\cos^2\theta}$$

$$= x\tan\theta - \frac{gx^2\sec^2\theta}{2V^2}$$

$$= x\tan\theta - \frac{gx^2}{2V^2}(1+\tan^2\theta)$$

(ii)



Applying the result from part (i),

when the particle clears the wall of height 2h metre

$$2h = a\tan\theta - \frac{ga^2}{2V^2}(1+\tan^2\theta)$$

$$\Rightarrow 2h - a\tan\theta = -\frac{ga^2}{2V^2}(1+\tan^2\theta) \quad \text{—① }$$

when the particle clears the wall of height h metre

$$h = b\tan\theta - \frac{gb^2}{2V^2}(1+\tan^2\theta)$$

$$\Rightarrow h - b\tan\theta = -\frac{gb^2}{2V^2}(1+\tan^2\theta) \quad \text{—②}$$

① ÷ ② $\quad \frac{2h - a\tan\theta}{h - b\tan\theta} = \frac{a^2}{b^2}$

$$\Rightarrow \tan\theta = \frac{2hb^2 - ha^2}{ab^2 - a^2b} = \frac{h(2b^2-a^2)}{ab(b-a)}$$